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# Mixed Hodge theory for unitary local systems

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## Introduction

Let $X$ be a compact complex manifold, bimeromorphic to a Kähler manifold. Let $D$ be a divisor in $X$ with local normal crossings, $U := X \setminus D$ and $V$ a local system on $U$. We assume that $V$ is unitary, i.e. $V$ has a positive definite hermitian form or equivalently, $V$ is given by a unitary representation of the fundamental group of $U$.

In [2], Deligne constructed the canonical extension $(\mathcal{M}, \nabla)$ of $V$. $\mathcal{M}$ is a holomorphic vector bundle on $X$ with a connection $\nabla$ with logarithmic poles along $D$ such that $\operatorname{Ker} \nabla|_U = V$. The hypercohomology of the logarithmic de Rham complex $(\Omega_X^{\cdot}\langle D \rangle \otimes \mathcal{M}, \nabla)$ is the cohomology of $V$ over $U$.

We want to show that there exists a very close connection between the topological cohomology of $V$ on $U$ and analytical (or algebraic) cohomology groups associated with $\mathcal{M}$. This connection is established by extending Deligne's construction of a mixed Hodge structure on $H^k(U, \mathbb{C})$ ([3]) to unitary local systems.

On the logarithmic de Rham complex, we construct two filtrations $F^{\cdot}$ and $W_{\cdot}$: $F^{\cdot}$ is the usual Hodge filtration which reflects the analytical properties, and $W_{\cdot}$ generalizes the weight filtration on $\Omega_X^{\cdot}\langle D \rangle$; the induced weight filtration on $H^k(U, V)$ is given in terms of purely topological data, namely the local cohomology of $V$ (along $D$) which one can describe very explicitly.

Let us now assume that $V$ and its hermitian form are induced by a real orthogonal local system $V_{\mathbb{R}}$, i.e.

$$V = V_{\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

and the hermitian form is the extension of the bilinear form on $V_{\mathbb{R}}$. Then we prove:

**Theorem.** *The induced Hodge and weight filtrations on $H^k(U, V)$ define a mixed $\mathbb{R}$-Hodge structure.*

If $V$ is defined over a noetherian subring $A$ of $\mathbb{R}$ such that $A \otimes \mathbb{Q}$ is a field, with the hermitian form defined over $A \otimes \mathbb{Q}$, one gets a mixed $A$-Hodge structure.

If $U$ is algebraic, then the mixed Hodge structure on $H^k(U, V)$ is independent of the choice of an algebraic compactification $j: U \hookrightarrow X$. It is functorial for maps $f: U' \to U$. For a finite unramified map $f: U \to U'$, the induced map

$$H^k(U, V) \to H^k(U', f_* V)$$

is a morphism of mixed Hodge structures.

Let us again drop the assumption that $V$ is defined over $\mathbb{R}$. Clearly one cannot expect a mixed Hodge structure on $H^k(U, V)$. But all properties of mixed Hodge structures that do not refer to the real structure carry over to this case. More precisely, we have:

**Theorem.** *Let $V$ be unitary.*

a) *The Hodge spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) \Rightarrow H^{p+q}(U, V)$$

*degenerates at $E_1$.*

b) *The Leray spectral sequence*

$$E_2^{p,q} = H^p(X, R^q j_* V) \Rightarrow H^{p+q}(U, V)$$

*degenerates at $E_3$.*

c) *The special sequence induced by the weight filtration on $\Omega_X^{\bullet}\langle D\rangle \otimes \mathcal{M}$*

$$E_1^{-m,k+m} = H^k(X, \mathrm{Gr}_m^{W_\cdot}(\Omega_X^{\bullet}\langle D\rangle \otimes \mathcal{M})) \Rightarrow H^k(X, \Omega_X^{\bullet}\langle D\rangle \otimes \mathcal{M})$$

*degenerates at $E_2$.*

d) *The conjugate linear isomorphism $H^k(U, V) \to H^k(U, V^\vee)$ respects the weight filtrations and induces conjugate linear isomorphisms between the "Hodge components"*

$$\mathrm{Gr}_{F^\cdot}^p \mathrm{Gr}_{p+q}^{W_\cdot} H^k(U, V) \text{ and } \mathrm{Gr}_{F^\cdot}^q \mathrm{Gr}_{p+q}^{W_\cdot} H^k(U, V^\vee).$$

The construction and investigation of the weight filtration is very close to Deligne's treatment of the case $V = \mathbb{C}_U$. Here we don't have to worry how the two filtrations pass to hypercohomology because of Deligne's fundamental theorem [4], 8.1.9. Classical Hodge theory is replaced by the pure Hodge structure on $H^k(X, j_* V)$ whose existence has been proved in [6]; the complex $\tilde{\Omega}_X^{\bullet}(\mathcal{M})$ defined in [6] agrees with $W_0(\Omega_X^{\bullet}\langle D\rangle \otimes \mathcal{M})$. For a unitary local system, $H^k(X, j_* V)$ is nothing but the intersection cohomology group $IH^k(X; V)$. The general reference for almost all facts needed on local systems and their canonical extensions is [2].

## § 1. The Hodge and weight filtrations

Let $X$ be any complex manifold, $D \subseteq X$ a local normal crossing divisor, $U := X \setminus D$, $j: U \hookrightarrow X$ the inclusion map, $V$ a local system on $U$, $(\mathcal{M}, \nabla)$ its canonical extension on $X$.

**(1. 1).** The Hodge filtration $F^{\cdot}$ of $\Omega_X^{\cdot}\langle D\rangle \otimes \mathcal{M}$ is the expected one: $F^p(\Omega_X^{\cdot}\langle D\rangle \otimes \mathcal{M})$ is the subcomplex of forms of degree $\geqq p$

$$F^p(\Omega_X^q\langle D\rangle \otimes \mathcal{M}) := \begin{Bmatrix} 0, & \text{if} \quad q<p \\ \Omega_X^q\langle D\rangle \otimes \mathcal{M}, & \text{if} \quad q \geqq p \end{Bmatrix}.$$

**(1. 2).** The definition of $W.(\Omega_X^{\cdot}\langle D\rangle \otimes \mathcal{M})$ is a little more involved. Recall the definition of $W.$ in case $V=\mathbb{C}$: $W_m(\Omega_X^{\cdot}\langle D\rangle)$ is the subcomplex of forms whose $(m+1)$-fold residue vanishes. So let us first define corresponding residue maps.

We start with introducing some notations that will be used throughout the rest of the paper:

For $m \geqq 1$, denote by $D_m$ the union of all $m$-fold intersections of distinct local components of $D$ and by $\tilde{D}_m$ the normalization of $D_m$. $\nu_m: \tilde{D}_m \to X$ is the composition of the normalization map $\tilde{D}_m \to D_m$ with the embedding $D_m \hookrightarrow X$. Let $\tilde{D}_0 := D_0 := X$, $\nu_0 := \mathrm{id}_X$. Let $\tilde{C}_m := \nu_m^{-1}(D_{m+1})$. $\tilde{C}_m$ is either empty or a normal crossing divisor in $\tilde{D}_m$.

On $\tilde{D}_m$, one has the (complex) local system $\tilde{\varepsilon}_m$ of orientations of the local components of $D$: For $\Delta \subseteq X$ such that $D \cap \Delta$ has smooth components and for a component $\tilde{Z}$ of $\nu_m^{-1}(\Delta)$, choose an order of the $m$ components of $D \cap \Delta$ whose intersection is $\nu_m(\tilde{Z})$. For two such open sets $\Delta^1, \Delta^2$, on every component of $\nu_m^{-1}(\Delta^1 \cap \Delta^2)$ the two orderings differ by a permutation, and the signs of these permutations are the transition constants of $\tilde{\varepsilon}_m$. Note that of course $\tilde{\varepsilon}_m$ is induced by a real orthogonal local system, defined even over $\mathbb{Z}$.

If the components of $D$ are smooth, then $\tilde{\varepsilon}_m$ is trivial.

There exist residue maps

$$\mathrm{Res}_m: \Omega_X^q\langle D\rangle \to \nu_{m*}(\Omega_{\tilde{D}_m}^{q-m}\langle \tilde{C}_m\rangle \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m)$$

commuting with $d$, thus giving a complex homomorphism

$$\mathrm{Res}_m: \Omega_X^{\cdot}\langle D\rangle \to \nu_{m*}(\Omega_{\tilde{D}_m}^{\cdot}\langle \tilde{C}_m\rangle \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m)\,[-m].$$

$\mathrm{Res}_m$ induces an isomorphism $R^m j_* \mathbb{C}_U \xrightarrow{\sim} \nu_{m*}\tilde{\varepsilon}_m$ (see [3], 3. 2. 8. 2).

Now assume that $V$ is unitary. We shall examine the restrictions of $j_* V$ to the strata $D_m \backslash D_{m+1}$, twisted with $\nu_{m*}\tilde{\varepsilon}_m$. Define

$$V_m := (j_* V_{|D_m \backslash D_{m+1}}) \underset{\mathbb{C}}{\otimes} (\nu_{m*}\tilde{\varepsilon}_{m|D_m \backslash D_{m+1}}).$$

**Proposition (1. 3).** a) $V_m$ *is a unitary local system on* $D_m \backslash D_{m+1}$.

b) *There exist a unique subvectorbundle* $\mathcal{M}_m$ *of* $\nu_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m$ *and a unique holomorphic integrable connection* $\nabla_m$ *on* $\mathcal{M}_m$ *with logarithmic poles along* $\tilde{C}_m$ *such that*

$$\mathrm{Ker}\,\nabla_{m|\tilde{D}_m \backslash \tilde{C}_m} = \nu_m^{-1} V_m.$$

c) *$(\mathcal{M}_m, \nabla_m)$ is the canonical extension of $v_m^{-1} V_m$.*

d) *There exists a unique subvectorbundle $\mathcal{M}_m^*$ of $v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m$ with*

$$v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m = \mathcal{M}_m \oplus \mathcal{M}_m^*$$

*such that for $\tilde{x} \in \tilde{D}_m \setminus \tilde{C}_m$*

$$\mathcal{M}_{m,\tilde{x}}^* = v_m^*(\{\sigma \in \mathcal{M}_{v_m(\tilde{x})} \mid (\sigma, \tau) = 0 \quad \text{for all} \quad \tau \in (j_* V)_{v_m(\tilde{x})}\}) \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_{m,\tilde{x}}.$$

**Remark.** In general, there will be no induced logarithmic connection on $v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m$ and no induced hermitian form on $v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_{m|\tilde{D}_m \setminus \tilde{C}_m}$.

*Proof.* The flat hermitian form on $V$ extends to a hermitian form on $j_* V$. So for the proof of a), we only have to show that $V_m$ is a local system on $D_m \setminus D_{m+1}$. The uniqueness assertions in b) and d) are clear, so it is enough to prove all four statements locally.

Let $x \in D$. Let $\Delta$ be a polycylinder around $x$ such that $D \cap \Delta = \bigcup_{j=1}^{s} D^j$ is a union of coordinate hyperplanes. Since $\pi_1(\Delta \setminus D)$ is abelian and $V$ is unitary, we can diagonalize all monodromy operators of $V_{|\Delta \setminus D}$ simultaneously. In other words, $V_{|\Delta \setminus D}$ is an orthogonal direct sum of unitary local systems of rank 1: $V_{|\Delta \setminus D} = \bigoplus_{i=1}^{r} V_{|\Delta \setminus D}^i$. If $\gamma_j$ is the monodromy transformation around $D^j$, then $\gamma_{j|V^i} = \gamma_{ij} \cdot \mathrm{id}_{V^i}$ with $\gamma_{ij} \in \mathbb{C}$, $|\gamma_{ij}| = 1$. We regard $V^i$ as a local system on $\Delta \setminus \bigcup_{\substack{j=1 \\ \gamma_{ij} \neq 1}}^{s} D^j$. $\mathcal{M}$ splits accordingly: $\mathcal{M} = \bigoplus_{i=1}^{r} \mathcal{M}^i$, where $\mathcal{M}^i$ has a logarithmic connection $\nabla^i$ with poles along $\bigcup_{\substack{j=1 \\ \gamma_{ij} \neq 1}}^{s} D^j$ such that $\operatorname{Ker} \nabla^i_{|\Delta \setminus \bigcup_{\substack{j=1 \\ \gamma_{ij} \neq 1}}^{s} D^j} = V^i$.

$(\mathcal{M}^i, \nabla^i)$ is the canonical extension of $V^i$, and $\nabla = \sum_{i=1}^{r} \nabla^i$.

a) If $y \in D_m \setminus D_{m+1}$, then there exist (uniquely determined) $j_1, \ldots, j_m$, $1 \leqq j_1 < \cdots < j_m \leqq s$, with $y \in D^{j_1} \cap \cdots \cap D^{j_m}$. Then for a small neighbourhood $\Delta_y$ of $y$,

$$j_* V_{|D_m \setminus D_{m+1} \cap \Delta_y} = \bigoplus_{\substack{i=1 \\ \gamma_{i,j_1} = \cdots = \gamma_{i,j_m} = 1}}^{r} V_{|D_m \setminus D_{m+1} \cap \Delta_y}^i,$$

so $j_* V_{|D_m \setminus D_{m+1}}$ is a local system near $y$.

b) Let $\tilde{Z}$ be a connected component of $v_m^{-1}(\Delta) \subseteq \tilde{D}_m$. Then $v_m(\tilde{Z}) = D^{j_1} \cap \cdots \cap D^{j_m}$ with $1 \leqq j_1 < \cdots < j_m \leqq s$. Then

$$v_m^{-1} V_{m|\tilde{Z} \setminus \tilde{C}_m} = v_m^{-1} \left( \bigoplus_{\substack{i=1 \\ \gamma_{i,j_1} = \cdots = \gamma_{i,j_m} = 1}}^{r} V_{|v_m(\tilde{Z})}^i \right) \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_{m|Z \setminus \tilde{C}_m}.$$

Clearly the subbundle of $v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_{m|\tilde{Z}}$ we are looking for is

$$\mathcal{M}_{m|\tilde{Z}} := v_m^* \Bigg( \bigoplus_{\substack{i=1 \\ \gamma_{i,j_1} = \cdots = \gamma_{i,j_m} = 1}}^{r} \mathcal{M}^i_{|v_m(\tilde{Z})} \Bigg) \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_{m|\tilde{Z}} .$$

Since $v_m(\tilde{Z})$ is not contained in the pole set of $\nabla^i$ if $\gamma_{i,j_1} = \cdots = \gamma_{i,j_m} = 1$, there is an induced logarithmic connection $\nabla_m$ on $\mathcal{M}_{m|\tilde{Z}}$ with poles along $\tilde{Z} \cap \tilde{C}_m$ and with

$$\operatorname{Ker} \nabla_{m|\tilde{Z} \setminus \tilde{C}_m} = v_m^{-1} V_{m|\tilde{Z} \setminus \tilde{C}_m} .$$

c) is clear from the description of $(\mathcal{M}_m, \nabla_m)$ in b).

d) Let $\tilde{Z}$ be the component of $\tilde{x}$ in $v_m^{-1}(\Delta) \subseteqq \tilde{D}_m$. Then $v_m(\tilde{Z}) = D^{j_1} \cap \cdots \cap D^{j_m}$, with $1 \leqq j_1 < \cdots < j_m \leqq s$. Clearly we have

$$\{\sigma \in \mathcal{M}_{v_m(\tilde{x})} \mid (\sigma, \tau) = 0 \text{ for all } \tau \in (j_* V)_{v_m(\tilde{x})}\}$$
$$= \Bigg( \bigoplus_{\substack{i=1 \\ \gamma_{i,j_\mu} \neq 1 \text{ for some } \mu}}^{r} \mathcal{M}^i \Bigg)_{v_m(\tilde{x})} ,$$

so define

$$\mathcal{M}^*_{m|\tilde{Z}} := v_m^* \Bigg( \bigoplus_{\substack{i=1 \\ \gamma_{i,j_\mu} \neq 1 \text{ for some } \mu}}^{r} \mathcal{M}^i_{|v_m(\tilde{Z})} \Bigg) \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_{m|\tilde{Z}} , \qquad \text{q.e.d.}$$

**(1. 4).** Now define $\operatorname{Res}_m(\mathcal{M})\colon \Omega_X^q \langle D \rangle \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \to v_{m*}(\Omega_{\tilde{D}_m}^{q-m} \langle \tilde{C}_m \rangle \underset{\mathcal{O}_{\tilde{D}_m}}{\otimes} \mathcal{M}_m)$ as the composition

$$\begin{aligned} \operatorname{Res}_m(\mathcal{M})\colon \Omega_X^q \langle D \rangle \underset{\mathcal{O}_X}{\otimes} \mathcal{M} &\to v_{m*}(\Omega_{\tilde{D}_m}^{q-m} \langle \tilde{C}_m \rangle \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m) \underset{\mathcal{O}_{\tilde{D}_m}}{\otimes} \mathcal{M} \\ &= v_{m*}(\Omega_{\tilde{D}_m}^{q-m} \langle \tilde{C}_m \rangle \underset{\mathcal{O}_{\tilde{D}_m}}{\otimes} (v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m)) \\ &\to v_{m*}(\Omega_{\tilde{D}_m}^{q-m} \langle \tilde{C}_m \rangle \underset{\mathcal{O}_{\tilde{D}_m}}{\otimes} \mathcal{M}_m), \end{aligned}$$

where the first map is $\operatorname{Res}_m \otimes \mathrm{id}$ and the second map is the identity on $\Omega_{\tilde{D}_m}^{q-m} \langle \tilde{C}_m \rangle$, tensored with the projection of $v_m^* \mathcal{M} \underset{\mathbb{C}}{\otimes} \tilde{\varepsilon}_m$ onto $\mathcal{M}_m$ along $\mathcal{M}_m^*$. Note that $\operatorname{Res}_m(\mathcal{M})$ is surjective.

**Lemma (1. 5).** $\operatorname{Res}_m(\mathcal{M}) \circ \nabla = \nabla_m \circ \operatorname{Res}_m(\mathcal{M})$, *i.e.* $\operatorname{Res}_m(\mathcal{M})$ *is a complex homomorphism*

$$\Omega_X^{\cdot} \langle D \rangle \otimes \mathcal{M} \to v_{m*}(\Omega_{\tilde{D}_m}^{\cdot} \langle \tilde{C}_m \rangle \otimes \mathcal{M}_m) [-m] .$$

*Proof.* We continue to use the local description given in the preceding proof. It suffices to show that for any $\tilde{x} \in \tilde{D}_m \backslash \tilde{C}_m$ and a germ $\sigma_i \in (\Omega_\Delta^q \langle D \cap \Delta \rangle \underset{\mathcal{O}_\Delta}{\otimes} \mathcal{M}^i)_{\nu_m(\tilde{x})}$, $\operatorname{Res}_m(\mathcal{M}^i)(\nabla^i(\sigma_i))_{\tilde{x}} = \nabla_m^i(\operatorname{Res}_m(\mathcal{M}^i)(\sigma_i))_{\tilde{x}}$. Let $\tilde{Z}$ be the component of $\nu_m^{-1}(\Delta)$ containing $\tilde{x}$ and $\nu_m(\tilde{Z}) = D^{j_1} \cap \cdots \cap D^{j_m}$, $1 \leqq j_1 < \cdots < j_m \leqq s$.

If there exists $\mu \in \{1, \ldots, m\}$ with $\gamma_{i, j_\mu} \neq 1$, then $\operatorname{Res}_m(\mathcal{M}^i)_{\tilde{x}} = 0$ and the assertion is trivial.

If $\gamma_{i, j_1} = \cdots = \gamma_{i, j_m} = 1$, then $V^i$ is a local system near $\nu_m(\tilde{x})$. Let $e_i$ be a local generator of $V^i$ near $\nu_m(\tilde{x})$. Then $\sigma_i = \tilde{\sigma}_i \otimes e_i$ and

$$\begin{aligned} \operatorname{Res}_m(\mathcal{M}^i) \circ \nabla^i(\tilde{\sigma}_i \otimes e_i) &= \operatorname{Res}_m(d\tilde{\sigma}_i) \otimes \nu_m^* e_i \\ &= d(\operatorname{Res}_m(\tilde{\sigma}_i)) \otimes \nu_m^* e_i \\ &= \nabla_m^i \circ \operatorname{Res}_m(\mathcal{M}^i)(\tilde{\sigma}_i \otimes e_i), \qquad \text{q.e.d.} \end{aligned}$$

We are now ready to give the definition of $W_\cdot(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M})$:

**Definition (1. 6).** For $m < 0$, set $W_m(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M}) := 0$. For $m \geqq 0$,

$$W_m(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M}) := \operatorname{Ker} \operatorname{Res}_{m+1}(\mathcal{M}).$$

Note that especially $W_0(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M}) = \tilde{\Omega}_X^\cdot(\mathcal{M})$ is the complex studied in [6].

From the preceding, we clearly have the following local description of $W_m(\Omega_X^q \langle D \rangle \otimes \mathcal{M})$ for a local system of rank 1: If $\Delta \subseteqq X$ is a polycylinder with coordinates $z_1, \ldots, z_n$ such that $D \cap \Delta$ is given by $z_1 \cdots z_s = 0$, and if $\gamma_j$ is the monodromy transformation of $V_{|\Delta \backslash D}$ around $\{z_j = 0\}$, then $W_m(\Omega_X^q \langle D \rangle \otimes \mathcal{M})_{|\Delta}$ is generated over $\mathcal{O}_\Delta$ by $\Omega_X^q \otimes \mathcal{M}_{|\Delta}$ and by all terms of the form

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \wedge \cdots \wedge dz_{j_q} \otimes \mu$$

where $\mu \in \Gamma(\Delta, \mathcal{M})$, $k \leqq q$, $1 \leqq j_1 < \cdots < j_k \leqq s < j_{k+1} < \cdots < j_q \leqq n$, and for at most $m$ indices $\kappa$ with $1 \leqq \kappa \leqq k$ we have $\gamma_{j_\kappa} = 1$.

**Proposition (1. 7).** a) *$W_\cdot(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M})$ is an increasing filtration.*

b) *$\operatorname{Res}_m(\mathcal{M})$ induces an isomorphism between*

$$\operatorname{Gr}_m^{W_\cdot}(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M}) \quad \text{and} \quad \nu_{m*}(\tilde{\Omega}_{\tilde{D}_m}^\cdot(\mathcal{M}_m))[-m].$$

*Proof.* As $\tilde{\Omega}_{\tilde{D}_m}^\cdot(\mathcal{M}_m) = \operatorname{Ker} \operatorname{Res}_1(\mathcal{M}_m)$, we have to show:

$$\operatorname{Ker}(\operatorname{Res}_{m+1}(\mathcal{M})) = \operatorname{Ker}(\nu_{m*}(\operatorname{Res}_1(\mathcal{M}_m)) \circ \operatorname{Res}_m(\mathcal{M})).$$

Let $\tilde{B}_m$ be the normalization of $\tilde{C}_m$ and $\mu_m : \tilde{B}_m \to X$ the composition of the normalization map $\tilde{B}_m \to \tilde{C}_m$, the embedding $\tilde{C}_m \hookrightarrow \tilde{D}_m$ and $\nu_m : \tilde{D}_m \to X$. Clearly $\mu_m = \nu_{m+1} \circ \varrho_m$ factors over $\nu_{m+1} : \tilde{D}_{m+1} \to X$, where $\varrho_m : \tilde{B}_m \to \tilde{D}_{m+1}$ is an isomorphism when restricted to any component of $\tilde{B}_m$ but in general maps several components of $\tilde{B}_m$ to the same component of $\tilde{D}_{m+1}$. Furthermore it is clear that $\varrho_m^*(\Omega_{\tilde{D}_{m+1}}^\cdot \langle \tilde{C}_{m+1} \rangle \otimes \mathcal{M}_{m+1})$ is isomorphic to $\Omega_{\tilde{B}_m}^\cdot \langle \mu_m^{-1}(D_{m+2}) \rangle \otimes (\mathcal{M}_m)_1$. Hence we have an induced injection $\nu_{m+1*}(\Omega_{\tilde{D}_{m+1}}^\cdot \langle \tilde{C}_{m+1} \rangle \otimes \mathcal{M}_{m+1}) \hookrightarrow \mu_{m*}(\Omega_{\tilde{B}_m}^\cdot \langle \mu_m^{-1}(D_{m+2}) \rangle \otimes (\mathcal{M}_m)_1)$.

It is straightforward to check from the definition, that the diagram

$$\begin{tikzcd}
\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M} \arrow[r, "\mathrm{Res}_m(\mathscr{M})"] \arrow[d, equal] & \nu_{m*}(\Omega_{\tilde{D}_m}^{\bullet}\langle \tilde{C}_m\rangle \otimes \mathscr{M}_m)[-m] \arrow[r, "\nu_{m*}(\mathrm{Res}_1(\mathscr{M}_m))"] & \mu_{m*}(\Omega_{\tilde{B}_m}^{\bullet}\langle \mu_m^{-1}(D_{m+2})\rangle \otimes (\mathscr{M}_m)_1)[-m-1] \\
\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M} \arrow[rr, "\mathrm{Res}_{m+1}(\mathscr{M})"'] & & \nu_{m+1*}(\Omega_{\tilde{D}_{m+1}}^{\bullet}\langle \tilde{C}_{m+1}\rangle \otimes \mathscr{M}_{m+1})[-m-1] \arrow[u]
\end{tikzcd}$$

is commutative, thereby finishing the proof of the proposition.

## § 2. Relations between the weight filtration and local cohomology

We now show that $\mathrm{Gr}_m^{W\cdot}(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M})$ is quasiisomorphic to $R^m j_* V[-m]$, thus exhibiting the topological nature of the induced weight filtration on the cohomology. Furthermore we see that $R^m j_* V$ is isomorphic to $j_{m*} V_m$ where $j_m\colon D_m \backslash D_{m+1} \hookrightarrow X$ is the inclusion map.

Define the canonical filtration $\tau.$ of $\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M}$ by:

$$\tau_m(\Omega_X^q\langle D\rangle \otimes \mathscr{M}) := \begin{Bmatrix} \Omega_X^q\langle D\rangle \otimes \mathscr{M}, & \text{if } q<m \\ \mathrm{Ker}\,\nabla \subseteq \Omega_X^q\langle D\rangle \otimes \mathscr{M}, & \text{if } q=m \\ 0, & \text{if } q>m \end{Bmatrix}.$$

**Proposition (2. 1).** *The inclusion map*

$$(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M}, \tau.) \to (\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M}, W.)$$

*is a quasiisomorphism of filtered complexes.*

*Proof.* We have to show that the induced maps

$$H^i(\mathrm{Gr}_m^{\tau.}(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M})) \to H^i(\mathrm{Gr}_m^{W\cdot}(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M}))$$

are isomorphisms.

Now

$$H^i(\mathrm{Gr}_m^{\tau.}(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M})) = \begin{Bmatrix} H^m(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M}) \cong R^m j_* V, & \text{if } i=m \\ 0, & \text{if } i \neq m \end{Bmatrix},$$

whereas

$$\begin{aligned} H^i(\mathrm{Gr}_m^{W\cdot}(\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M})) &\cong H^i(\nu_{m*}(\tilde{\Omega}_{\tilde{D}_m}^{\bullet}(\mathscr{M}_m))[-m]) \\ &\cong \nu_{m*} H^{i-m}(\tilde{\Omega}_{\tilde{D}_m}^{\bullet}(\mathscr{M}_m)) \\ &\cong \begin{Bmatrix} j_{m*} V_m, & \text{if } i=m \\ 0, & \text{if } i \neq m \end{Bmatrix}; \end{aligned}$$

here $j_m: D_m \setminus D_{m+1} \hookrightarrow X$ is the inclusion map. For the last equality, we used that the inclusion map $\tilde{j}_{m*}(v_m^{-1} V_m) \to \tilde{\Omega}^{\cdot}_{\tilde{D}_m}(\mathcal{M}_m)$ is a quasiisomorphism where $\tilde{j}_m: \tilde{D}_m \setminus \tilde{C}_m \hookrightarrow \tilde{D}_m$ is the inclusion map (see [6]). Hence we have to show that

$$\mathrm{Res}_m(\mathcal{M}): H^m(\Omega^{\cdot}_X\langle D\rangle \otimes \mathcal{M}) \to j_{m*} V_m = v_{m*} \tilde{j}_{m*}(v_m^{-1} V_m)$$

is an isomorphism.

For this, we again use the local description given in the proof of (1. 3). Clearly we may replace $V$ by one of the $V^i$, i.e. we may assume that $V$ has rank 1. Let $x \in D$. We distinguish between two cases:

*Case* 1. $j_* V$ is a local system near $x$. Then of course $j_* V$ is trivial near $x$, and we are done by the classical isomorphism

$$\mathrm{Res}_m: H^m(\Omega^{\cdot}_X\langle D\rangle) \xrightarrow{\sim} j_{m*} \mathbb{C}_{D_m \setminus D_{m+1}} = v_{m*} \mathbb{C}_{\tilde{D}_m}.$$

*Case* 2. There exists a local component $D^j$ of $D$ through $x$ with $\gamma_j \neq \mathrm{id}_V$. Let us first show that $(j_{m*} V_m)_x = 0$:

Let $\tilde{Z}$ be any component of $v_m^{-1}(\Delta)$ with $x \in v_m(\tilde{Z})$. Then $v_m(\tilde{Z}) = D^{j_1} \cap \cdots \cap D^{j_m}$, $1 \leqq j_1 < \cdots < j_m \leqq s$. Either there exists $\mu \in \{1, \ldots, m\}$ with $\gamma_{j_\mu} \neq \mathrm{id}$, in which case $\mathcal{M}_{m|\tilde{Z}} = 0$, or $\gamma_{j_1} = \cdots = \gamma_{j_m} = \mathrm{id}$; but as $\gamma_j \neq \mathrm{id}$, $\nabla_m$ is not holomorphic at $\tilde{x} \in \tilde{Z}$, $v_m(\tilde{x}) = x$, and therefore $\tilde{j}_{m*}(v_m^{-1} V_m)_{\tilde{x}} = 0$. Hence, $(j_{m*} V_m)_x = (v_{m*} \tilde{j}_{m*}(v_m^{-1} V_m))_x = 0$.

Finally we show $(R^m j_* V)_x = 0$. It suffices to prove: For all small polycylinders $\Delta' \subseteqq \Delta$ centered at $x$ such that $D \cap \Delta'$ is a union of coordinate hyperplanes,

$$H^m(\Delta' \setminus D, V_{|\Delta' \setminus D}) = 0.$$

We can write $\Delta' = \Delta'_1 \times \Delta'_2$, with projections $p_1: \Delta' \to \Delta'_1$, $p_2: \Delta' \to \Delta'_2$, such that $D \cap \Delta' = p_1^* D'_1 + p_2^* D'_2$ with normal crossing divisors $D'_1 \subseteqq \Delta'_1$, $D'_2 \subseteqq \Delta'_2$ and $p_1^* D'_1$ is the union of all components $D^j \cap \Delta'$ of $D \cap \Delta'$ with $\gamma_j \neq \mathrm{id}$. Then $V_{|\Delta'} = p_1^{-1} V'_1$ where $V'_1$ is a local system on $\Delta'_1 \setminus D'_1$ with nontrivial monodromy around every component of $D'_1$. Note that by assumption $x \in p_1^{-1}(D'_1)$. Now the Künneth formula

$$H^m(\Delta' \setminus D, V_{|\Delta' \setminus D}) \cong \bigoplus_{p+q=m} H^p(\Delta'_1 \setminus D'_1, V'_1) \otimes_{\mathbb{C}} H^q(\Delta'_2 \setminus D'_2, \mathbb{C})$$

shows that it is enough to have $H^p(\Delta' \setminus D'_1, V'_1) = 0$ for all $p \geqq 0$, i.e. $(R^p j'_{1*} V'_1)_{p_1(x)} = 0$, $j'_1: \Delta'_1 \setminus D'_1 \hookrightarrow \Delta'_1$ the inclusion map. This is clear for $p = 0$, and for $p > 0$ it follows from $\Omega^{\cdot}_{\Delta'_1}\langle D'_1\rangle \otimes \mathcal{M}'_1 = \tilde{\Omega}^{\cdot}_{\Delta'_1}(\mathcal{M}'_1)$, $\mathcal{M}'_1$ the canonical extension of $V'_1$, and the fact that $\tilde{\Omega}^{\cdot}_{\Delta'_1}(\mathcal{M}'_1)$ is exact at level $p$, q.e.d.

**Corollary (2. 2).** *$H^m(\mathrm{Gr}^W_m \cdot (\Omega^{\cdot}_X\langle D\rangle \otimes \mathcal{M})) \cong R^m j_* V \cong j_{m*} V_m$, where the second isomorphism is induced by* $\mathrm{Res}_m(\mathcal{M})$.

**Remark.** Of course, the inclusion map

$$(\Omega^{\cdot}_X\langle D\rangle \otimes \mathcal{M}, F^{\cdot}, \tau.) \to (\Omega^{\cdot}_X\langle D\rangle \otimes \mathcal{M}, F^{\cdot}, W.)$$

is in general *not* a quasiisomorphism of bifiltered complexes.

## § 3. Unitary local systems defined over a subring of $\mathbb{C}$

Let $A$ be a noetherian subring of $\mathbb{C}$, closed under conjugation, such that $A \otimes \mathbb{Q}$ is a field.

**Definition (3. 1).** An $A$-unitary local system is a complex unitary local system $V$ such that there exist a local system $V_A$ over $A$ (i.e. a sheaf of $A$-modules which is locally isomorphic to the constant sheaf $A^r$ for some $r \in \mathbb{N}$) with $V = V_A \underset{A}{\otimes} \mathbb{C}$ and a hermitian symmetric $A \otimes \mathbb{Q}$-valued positive definite form on $V_A$ (or on $V_{A\otimes\mathbb{Q}} := V_A \otimes \mathbb{Q}$), which induces the hermitian form on $V$.

For example, if $V$ is given by a representation

$$\pi_1(U) \to GL(r, A) \cap U(r, \mathbb{C}) \subseteqq GL(r, \mathbb{C}),$$

then $V$ is $A$-unitary.

Notice that for an $A$-unitary local system $V$, the local systems $V_m$ are canonically $A$-unitary:

$$V_{m,A} := V_{A,m} := (j_* V_{A|D_m \setminus D_{m+1}}) \underset{\mathbb{Z}}{\otimes} (v_{m*} \tilde{\varepsilon}_{m,\mathbb{Z}|D_m \setminus D_{m+1}})$$

where $\tilde{\varepsilon}_m$ is $\mathbb{Z}$-unitary by definition.

In contrast to the case $V = \mathbb{C}_U$, $R^m j_* V_A$ may have torsion: As a trivial example, take $A = \mathbb{Z}$, $X = \mathbb{P}^1$, $U = \mathbb{C}^*$, $V_{\mathbb{Z}}$ the local system of rank 1 over $\mathbb{Z}$ on $U$ given by the representation of $\pi_1(U) \cong \mathbb{Z}$ which maps the class of a loop running counterclockwise once around 0 to $-\mathrm{id}_{\mathbb{Z}}$, then clearly $(R^1 j_* V_{\mathbb{Z}})_0 \cong (R^1 j_* V_{\mathbb{Z}})_\infty \cong \mathbb{Z}/2\mathbb{Z}$.

Replacing $A$ by $A \otimes \mathbb{Q}$ in order to avoid problems with torsion in $R^m j_* V_A$, we now want to compare the $A \otimes \mathbb{Q}$-structures on $R^m j_* V = R^m j_* V_{A\otimes\mathbb{Q}} \otimes \mathbb{C}$ and $j_{m*} V_m = j_{m*} V_{m, A\otimes\mathbb{Q}} \otimes \mathbb{C}$ via the isomorphism in (2. 2). The result is:

**Lemma (3. 2).** *The isomorphism $R^m j_* V \xrightarrow{\sim} j_{m*} V_m$ in (2. 2) maps $R^m j_* V_{A\otimes\mathbb{Q}}$ to*

$$j_{m*} V_{m, A\otimes\mathbb{Q}}(-m) := (2\pi i)^{-m} j_{m*} V_{m, A\otimes\mathbb{Q}} \subseteqq j_{m*} V_m.$$

*Proof.* Let $x \in D$. Then in a small neighbourhood $\Delta$ of $x$, $V_{A\otimes\mathbb{Q}|\Delta\setminus D}$ splits $V_{A\otimes\mathbb{Q}|\Delta\setminus D} = V'_{A\otimes\mathbb{Q}} \oplus V''_{A\otimes\mathbb{Q}}$, such that $j_* V'_{A\otimes\mathbb{Q}}$ is a (trivial) local system over $A \otimes \mathbb{Q}$ near $x$ and $(j_* V''_{A\otimes\mathbb{Q}})_x = 0$. $V_{|\Delta\setminus D} = V' \oplus V''$ splits accordingly. Then the proof of (2. 1) shows that $(R^m j_* V'')_x = 0 = (j_{m*} V''_m)_x$, hence $(R^m j_* V''_{A\otimes\mathbb{Q}})_x = 0 = (j_{m*} V''_{m, A\otimes\mathbb{Q}})_x$ since we have no torsion.

So we have to show the lemma only for $V'_{A\otimes\mathbb{Q}}$. But the case of a trivial local system is proved in [3], 3. 1. 9.

## § 4. Functoriality

Let us consider the functorial behaviour of the Hodge and weight filtrations.

Clearly they are functorial for maps $\varphi: V \to V'$ between unitary local systems on $U$ which respect the hermitian forms.

Not so entirely obvious is the functoriality for holomorphic maps $\bar{f}: (X', U') \to (X, U)$. The problem is that in general the lift of the canonical extension is not the canonical extension of the lifted local system. Furthermore it might happen (say for $V$ with rank $V=1$) that $D' := X' \setminus U'$ has a component $D_1'$ across which $f^{-1}V (f := \bar{f}_{|U'})$ extends to a local system, but at the same time $D_1'$ lies over a component $D_1$ of $D := X \setminus U$ along which $V$ is not trivial; then a form in $W_0$ may have poles along $D_1$, but has to be regular along $D_1'$. But it turns out that the lift of the canonical extension is "smaller" than the canonical extension of the lifted local system, and the "difference" takes care of the unwanted poles.

**Proposition (4. 1).** *Let $\bar{f}: X' \to X$ be a holomorphic map, $D' \subseteqq X'$ and $D \subseteqq X$ local normal crossing divisors with $\bar{f}^{-1}(D) \subseteqq D'$, $U' := X' \setminus D'$, $U := X \setminus D$, $f: U' \to U$ the restricted map. Let $V$ be a local system on $U$, $\mathcal{M}$ the canonical extension of $V$ on $X$ and $\mathcal{M}'$ the canonical extension of $f^{-1}V$ on $X'$.*

a) *There exists a unique injection $\bar{f}^*\mathcal{M} \hookrightarrow \mathcal{M}'$ such that the diagram*

$$\begin{tikzcd} \bar{f}^*\mathcal{M}_{|U'} \arrow[rr, hook] & & \mathcal{M}'_{|U'} \\ & f^{-1}V \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

*commutes.*

b) *If $V$ is unitary, then the induced map*

$$\bar{f}^{-1}(\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M}) \to \Omega_{X'}^\bullet\langle D'\rangle \otimes \mathcal{M}'$$

*respects both the Hodge and weight filtrations.*

*Proof.* Since the uniqueness in a) is clear, it is enough to prove (4. 1) locally.

So let $x' \in D$, $\Delta'$ a polycylinder around $x'$ with coordinates $(w_1, \ldots, w_{n'})$, $\Delta$ a polycylinder around $\bar{f}(x')$ with coordinates $(z_1, \ldots, z_n)$, $\bar{f}(\Delta') \subseteqq \Delta$, $D' \cap \Delta'$ given by $w_1 \cdots w_{s'} = 0$, $D \cap \Delta$ given by $z_1 \cdots z_s = 0$. Then $\bar{f}^* z_j = \varepsilon_j \cdot \prod_{i=1}^{s'} w_i^{a_{ij}}$ for $1 \leqq j \leqq s$, with units $\varepsilon_j$ and $a_{ij} \geqq 0$. Let $\varrho_i'$ be a loop in $\Delta' \setminus D'$ running counterclockwise once around $\{w_i = 0\}$ $(i = 1, \ldots, s')$, and $\varrho_j$ a loop in $\Delta \setminus D$ running counterclockwise once around $\{z_j = 0\}$ $(j = 1, \ldots, s)$. Then $f_*: \pi_1(\Delta' \setminus D') \to \pi_1(\Delta \setminus D)$ maps (the class of) $\varrho_i'$ to (the class of) $\sum_{j=1}^{s} a_{ij} \varrho_j$.

Let $\Gamma_j \in GL(r, \mathbb{C})$ be the monodromy transformation associated to $\varrho_j$ $(j = 1, \ldots, s)$. Let $B_1, \ldots, B_s$ be commuting $r \times r$-matrices with eigenvalues in the strip $\{0 \leqq \operatorname{Re} \lambda < 1\}$ and $\exp(-2\pi i B_j) = \Gamma_j$.

a) We have $\mathcal{M}_{|\Delta} \cong \mathcal{O}_\Delta^r$ with $\nabla_{\mathcal{M}}$ given by

$$\nabla_{\mathcal{M}} g = dg + \sum_{j=1}^{s} B_j g \cdot \frac{dz_j}{z_j}$$

for an $r$-tuple $g$ of holomorphic functions.

Hence $\bar{f}^*\mathcal{M}_{|\Delta'} \cong \mathcal{O}^r_{\Delta'}$ with the lifted connection $\nabla_{\bar{f}^*\mathcal{M}}$ given by

$$\nabla_{\bar{f}^*\mathcal{M}} g' = d g' + \sum_{j=1}^{s} B_j g' \cdot \frac{d\varepsilon_j}{\varepsilon_j} + \sum_{i=1}^{s'} \left( \sum_{j=1}^{s} a_{ij} B_j \right) g' \cdot \frac{d w_i}{w_i}.$$

After an appropriate change of the isomorphism $\bar{f}^*\mathcal{M}_{|\Delta'} \cong \mathcal{O}^r_{\Delta'}$, we can clearly get $\nabla_{\bar{f}^*\mathcal{M}}$ in the form

$$\nabla_{\bar{f}^*\mathcal{M}} g' = d g' + \sum_{i=1}^{s'} \left( \sum_{j=1}^{s} a_{ij} B_j \right) g' \cdot \frac{d w_i}{w_i}.$$

From the above description of $f_*\colon \pi_1(\Delta'\backslash D') \to \pi_1(\Delta\backslash D)$ we see that $\nabla_{\mathcal{M}'}$ on $\mathcal{M}'_{|\Delta'} \cong \mathcal{O}^r_{\Delta'}$ is given by

$$\nabla_{\mathcal{M}'} \tilde{g} = d\tilde{g} + \sum_{i=1}^{s'} B'_i \tilde{g} \cdot \frac{d w_i}{w_i}$$

where $B'_1, \dots, B'_s$ are commuting matrices with eigenvalues in the strip $\{0 \leqq \operatorname{Re}\lambda < 1\}$ and

$$\exp(-2\pi i B'_i) = \prod_{j=1}^{s} \Gamma_j^{a_{ij}} = \exp\left( -2\pi i \sum_{j=1}^{s} a_{ij} B_j \right).$$

Hence $C_i := \sum_{j=1}^{s} a_{ij} B_j - B'_i$ are commuting semisimple matrices whose eigenvalues are nonnegative integers.

The desired injection $\bar{f}^*\mathcal{M}_{|\Delta'} \cong \mathcal{O}^r_{\Delta'} \hookrightarrow \mathcal{O}^r_{\Delta'} \cong \mathcal{M}'_{|\Delta'}$ extending the identity map on flat sections is

$$\mathcal{O}^r_{\Delta'} \to \mathcal{O}^r_{\Delta'} \qquad g' \mapsto \tilde{g} = \prod_{i=1}^{s'} w_i^{C_i} \cdot g'.$$

b) Compatibility with the Hodge filtration is trivial, so let us consider the weight filtrations. Clearly we may assume that $V$ has rank 1.

It will be convenient to use the description of $W_m(\Omega^q_X\langle D\rangle \otimes \mathcal{M})$ in terms of local coordinates which was given after its definition in (1.6): $W_m(\Omega^q_X\langle D\rangle \otimes \mathcal{M})_{|\Delta}$ is generated over $\mathcal{O}_\Delta$ by $\Omega^q_X \otimes \mathcal{M}_{|\Delta}$ and all expressions of the form

$$\frac{d z_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{d z_{j_k}}{z_{j_k}} \wedge d z_{j_{k+1}} \wedge \cdots \wedge d z_{j_q} \otimes \mu,$$

where $\mu \in \Gamma(\Delta, \mathcal{M})$, $1 \leqq j_1 < \cdots < j_k \leqq s < j_{k+1} < \cdots < j_q \leqq n$, and for at most $m$ indices $\kappa$ with $1 \leqq \kappa \leqq k$ we have $\Gamma_{j_\kappa} = 1$ $(\in \mathbb{C})$.

Using trivializations of $\mathcal{M}_{|\Delta}$ and $\mathcal{M}'_{|\Delta'}$ as above, the image under $\bar{f}^{-1}(\Omega^\bullet_X\langle D\rangle \otimes \mathcal{M}) \to \Omega^\bullet_{X'}\langle D'\rangle \otimes \mathcal{M}'$ of the above element, lifted to $\Delta'$, is:

$$\prod_{i=1}^{s'} w_i^{C_i} \cdot \bigwedge_{\kappa=1}^{k} \left( \frac{d\varepsilon_{j_\kappa}}{\varepsilon_{j_\kappa}} + \sum_{i_\kappa=1}^{s'} a_{i_\kappa, j_\kappa} \frac{d w_{i_\kappa}}{w_{i_\kappa}} \right) \wedge \bar{f}^*(d z_{j_{k+1}} \wedge \cdots \wedge d z_{j_q}) \otimes \mu',$$

with $\mu' \in \Gamma(\Delta', \mathcal{M}')$.

Expanding this expression, we get a sum of terms of the type

$$\prod_{i=1}^{s'} w_i^{C_i} \cdot a_{i_1,j_1} \cdots a_{i_{k'},j_{k'}} \cdot \frac{d w_{i_1}}{w_{i_1}} \wedge \cdots \wedge \frac{d w_{i_{k'}}}{w_{i_{k'}}} \wedge \sigma$$

(modulo renumbering, of course) with $k' \leqq k$, $\sigma$ a holomorphic $\mathcal{M}'$-valued form.

Let us take $\kappa' \in \{1, \ldots, k'\}$. If $C_{i_{\kappa'}} > 0$, then the term has no pole along $\{w_{i_{\kappa'}} = 0\}$. If $C_{i_{\kappa'}} = 0$ and $\Gamma'_{i_{\kappa'}} = 1$ ($\Gamma'_i$ is the monodromy transformation associated to $\varrho'_i$), then from $0 = B'_{i_{\kappa'}} = \sum_{j=1}^{s} a_{i_{\kappa'},j} B_j$ we see that in case $a_{i_1,j_1} \neq 0, \ldots, a_{i_{k'},j_{k'}} \neq 0$ we must have $B_{j_{\kappa'}} = 0$, i.e. $\Gamma_{j_{\kappa'}} = 1$. Hence there are at most $m$ indices of that type, q.e.d.

## § 5. Hodge decomposition and duality theorems for the cohomology of $j_* V$; relations with intersection cohomology

We state the main result in [6] and derive from the proof in [6] Poincaré and Serre duality for the cohomology of $j_* V$ and $\tilde{\Omega}^p_X(\mathcal{M})$. With their help we extend the Hodge decomposition to compact manifolds bimeromorphic to Kähler manifolds.

Denote by $F^\cdot$ the Hodge filtration on $H^k(X, j_* V)$, induced by the Hodge filtration on $\tilde{\Omega}^\cdot_X(\mathcal{M})$. Let $\bar{F}^\cdot$ be the lift under the conjugation map $H^k(X, j_* V) \to H^k(X, j_* V^\vee)$ of the Hodge filtration on $H^k(X, j_* V^\vee)$, and let

$$H^{p,q}(X, j_* V) := (F^p \cap \bar{F}^p)\, H^{p+q}(X, j_* V).$$

**Theorem (5. 1).** *Let $X$ be a compact Kähler manifold, $D \subseteqq X$ a local normal crossing divisor, $U := X \backslash D$, $j\colon U \hookrightarrow X$ the inclusion map, $V$ a unitary local system on $U$, $\mathcal{M}$ the canonical extension of $V$.*

a) *The spectral sequence*

$$E_1^{p,q} = H^q(X, \tilde{\Omega}^p_X(\mathcal{M})) \Rightarrow H^{p+q}(X, j_* V)$$

*degenerates at $E_1$.*

b) *The maps $H^{p,q}(X, j_* V) \xrightarrow{\sim} H^q(X, \tilde{\Omega}^p_X(\mathcal{M}))$ are isomorphisms, and*

$$H^k(X, j_* V) = \bigoplus_{p+q=k} H^{p,q}(X, j_* V).$$

c) *The conjugation map induces conjugate linear isomorphisms*

$$H^q(X, \tilde{\Omega}^p_X(\mathcal{M})) \xrightarrow{\sim} H^p(X, \tilde{\Omega}^q_X(\mathcal{N}))$$

*where $\mathcal{N}$ is the canonical extension of $V^\vee$.*

The proof in [6] is by identifying $\tilde{\Omega}^p_X(\mathcal{M})$ as the sheaf of locally square-integrable holomorphic $p$-forms with values in $V$ with respect to a suitable metric with poles along $D$. Then $H^q(X, \tilde{\Omega}^p_X(\mathcal{M}))$ and $H^{p+q}(X, j_* V)$ are identified with $L^2$-cohomology groups which in turn are isomorphic to spaces of harmonic forms. Then one uses the Kähler identities as in the classical case to get the decomposition.

Note that without the Kähler hypothesis one still can represent the cohomology groups by harmonic forms on which the $*$-operator acts; as in the classical case, one concludes Poincaré duality and Serre duality.

**Theorem (5. 2).** *Let $X$ be a compact manifold, $D \subseteqq X$ a local normal crossing divisor, $U := X \backslash D$, $j: U \hookrightarrow X$ the inclusion map, $V$ a unitary local system on $U$, $\mathcal{M}$, $\mathcal{N}$ the canonical extensions of $V$, $V^\vee$, resp.*

a) *Poincaré duality*:

$$H^k(X, j_* V) \underset{\mathbb{C}}{\otimes} H^{2n-k}(X, j_* V^\vee) \to H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$$

*is a perfect pairing.*

b) *Serre duality*:

$$H^p(X, \tilde{\Omega}_X^q(\mathcal{M})) \underset{\mathbb{C}}{\otimes} H^{n-p}(X, \tilde{\Omega}_X^{n-q}(\mathcal{N})) \to H^n(X, \Omega_X^n) \cong \mathbb{C}$$

*is a perfect pairing.*

Note that the map $\tilde{\Omega}_X^q(\mathcal{M}) \otimes \tilde{\Omega}_X^{n-q}(\mathcal{N}) \to \Omega_X^n\langle D\rangle$ actually has its image in $\Omega_X^n$:

We only have to prove this locally, so we may assume that $\operatorname{rk} V = 1$. If $D'$ is the maximal reduced subdivisor of $D$ such that $j_* V_{|X \backslash D'}$ is a local system, then $\tilde{\Omega}_X^q(\mathcal{M}) = \Omega_X^q\langle D'\rangle \otimes \mathcal{M}$ and $\tilde{\Omega}_X^{n-q}(\mathcal{N}) = \Omega_X^{n-q}\langle D'\rangle \otimes \mathcal{N}$. Furthermore

$$\mathcal{N} = \mathcal{M}^\vee \otimes \mathcal{O}_X(-D'),$$

and the above map is given by

$$\begin{aligned}\tilde{\Omega}_X^q(\mathcal{M}) \otimes \tilde{\Omega}_X^{n-q}(\mathcal{N}) &= \Omega_X^q\langle D'\rangle \otimes \mathcal{M} \otimes \Omega_X^{n-q}\langle D'\rangle \otimes \mathcal{M}^\vee \otimes \mathcal{O}_X(-D') \\ &\to \Omega_X^n\langle D'\rangle \otimes \mathcal{O}_X(-D') = \Omega_X^n.\end{aligned}$$

With the help of (5. 2), we can easily extend (5. 1) to manifolds bimeromorphic to Kähler manifolds:

**Corollary (5. 3).** (5. 1) *remains true if $X$ is only bimeromorphic to a Kähler manifold.*

The *proof* is an adaption of [1], Prop. (5. 3): Let $\bar{f}: X' \to X$ be a modification with a compact Kähler manifold $X'$ such that $D' := \bar{f}^{-1}(D)$ is a (local) normal crossing divisor, $U' := X' \backslash D'$, $f := \bar{f}_{|U'}: U' \to U$, $V' := f^{-1} V$, $j': U' \hookrightarrow X'$ the inclusion map. From (4. 1), we get a commutative diagram

$$\begin{tikzcd} H^q(X, \tilde{\Omega}_X^p(\mathcal{M})) \otimes H^{n-q}(X, \tilde{\Omega}_X^{n-p}(\mathcal{N})) \arrow[r] \arrow[d] & H^n(X, \Omega_X^n) \cong \mathbb{C} \arrow[d, "\wr"] \\ H^q(X', \tilde{\Omega}_{X'}^p(\mathcal{M}')) \otimes H^{n-q}(X', \tilde{\Omega}_{X'}^{n-p}(\mathcal{N}')) \arrow[r] & H^n(X', \Omega_{X'}^n) \cong \mathbb{C} \end{tikzcd}$$

where $\mathscr{M}'$, $\mathscr{N}'$ are the canonical extensions of $V'$, $V'^{\vee}$, resp. From Serre duality (5. 2) it follows that $H^q(X, \tilde{\Omega}^p_X(\mathscr{M})) \to H^q(X', \tilde{\Omega}^p_{X'}(\mathscr{M}'))$ is injective. Hence the Hodge spectral sequence of $V$ injects into the degenerating Hodge spectral sequence of $V'$ and thus degenerates at $E_1$, too.

As above we infer from Poincaré duality (5. 2) that $H^k(X, j_* V) \to H^k(X', j'_* V')$ is injective. The map clearly respects $F^{\cdot}$ and $\bar{F}^{\cdot}$, so from $(F^p \cap \bar{F}^{k-p+1})(H^k(X', j'_* V')) = 0$ we deduce $(F^p \cap \bar{F}^{k-p+1})(H^k(X, j_* V)) = 0$. Counting dimensions as in [1], (5. 3. 7) we see that $H^k(X, V) = F^p H^k(X, j_* V) \oplus \bar{F}^{k-p+1} H^k(X, j_* V)$. Hence $H^q(X, \tilde{\Omega}^p_X(\mathscr{M}))$ is isomorphic to $H^{p,q}(X, j_* V)$, and $H^{p+q}(X, j_* V) = \bigoplus_{p+q=k} H^{p,q}(X, j_* V)$, q.e.d.

**Remarks.** a) Clearly, in the situation of (5. 1) or (5. 3), Poincaré duality and Serre duality agree on $H^k(X, j_* V) = \bigoplus_{p+q=k} H^q(X, \tilde{\Omega}^p_X(\mathscr{M}))$.

b) If $V$ is $A$-unitary with $A \subseteq \mathbb{R}$, then (5. 1) and (5. 3) just say that the Hodge filtration defines a pure Hodge structure of weight $k$ on $H^k(X, j_* V)$.

As the referee pointed out, it would be interesting to establish the connection with intersection cohomology (see [5]). In fact, we have:

**Proposition (5. 4).** a) $D_X(j_* V) = j_* V^{\vee}[2n]$, *where $D_X$ is the Verdier duality functor on $X$.*

b) $j_* V[2n]$ *is the (middle) intersection complex of $V$; consequently,*

$$H^k(X, j_* V) \cong IH^k(X; V).$$

c) *The pairing in part* a) *of* (5. 2) *agrees with the Poincaré duality pairing in intersection cohomology.*

*Proof.* Using the axiomatic characterization of the intersection complex in [5], parts b) and c) follow trivially from a), so let us prove a). We have a canonical morphism

$$D_X(j_* V) \to j_* j^* D_X(j_* V) = j_* D_U(j^! j_* V) = j_* D_U V = j_* V^{\vee}[2n],$$

so we may prove a) locally and may therefore assume $\operatorname{rk} V = 1$. We replace $U$ by the maximal open subset of $X$ to which $V$ extends to a local system. As we have seen before (see the proof of (2. 1)), we then have $Rj_* V = j_* V = j_! V$, and therefore $j_* V = Rj_! V$. By the same reasoning, $j_* V^{\vee} = Rj_* V^{\vee}$, and hence

$$D_X(j_* V) = D_X(Rj_! V) = Rj_*(D_U V) = Rj_* V^{\vee}[2n] = j_* V^{\vee}[2n], \text{ q.e.d.}$$

## § 6. Mixed Hodge structures on $H^k(U, V_A)$ for real $A$

Let us now define the Hodge and weight filtrations on $H^k(U, V)$:

**Definition (6. 1).**

$$F^p H^k(U, V) := \operatorname{Im}(\mathbb{H}^k(X, F^p(\Omega^{\cdot}_X\langle D\rangle \otimes \mathscr{M})) \to \mathbb{H}^k(X, \Omega^{\cdot}_X\langle D\rangle \otimes \mathscr{M}) = H^k(U, V));$$

we shall soon see that this map is injective.

For the weight filtration, one uses a shift by $k$:

$$W_{m+k}H^k(U, V) := \mathrm{Im}\,(\mathbb{H}^k(X, W_m(\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M})) \to \mathbb{H}^k(X, \Omega_X^\bullet\langle D\rangle \otimes \mathcal{M}) = H^k(U, V)).$$

The following lemma is an immediate consequence of the definitions and the description of the weight filtration on $\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M}$:

**Lemma (6. 2).** a) $W_i H^k(U, V) = 0$ *for* $i < k$.

b) $W_k H^k(U, V) = \mathrm{Im}\,(H^k(X, j_* V) \to H^k(U, V))$.

c) *Let $m_0$ be the largest $m \in \mathbb{N}$ such that there exist $x \in X$ and $m$ local components of $D$ through $x$ such that the monodromy of $V$ around these components has an eigenvalue 1. Then*

$$W_{k+m_0} H^k(U, V) = H^k(U, V).$$

We now assume that $V$ is $A$-unitary with $A \subseteqq \mathbb{R}$. Our aim is to prove:

**Theorem (6. 3).** *Let $X$ be bimeromorphic to a compact Kähler manifold, $D \subseteqq X$ a local normal crossing divisor, $U := X \backslash D$. Let $V$ be an $A$-unitary local system on $U$, with a subring $A \subseteqq \mathbb{R}$. Then the Hodge and weight filtrations on $H^k(U, V)$ give a mixed Hodge structure on $H^k(U, V_A)$.*

By Deligne's analysis of bifiltered complexes ([3], 3. 2, [4], 8. 1. 9) we have to show (for the notation, see [4], 8. 1):

**Proposition (6. 4).**

$$(Rj_* V_A, (Rj_* V_{A\otimes\mathbb{Q}}, \tau.), (\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M}, F^\bullet, W.))$$

*is a cohomological mixed $A$-Hodge complex.*

*Proof.* The quasiisomorphism $(Rj_* V_{A\otimes\mathbb{Q}}, \tau.) \otimes \mathbb{C} \to (\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M}, W.)$ is the one given in (2. 1). So it remains to show:

$$(\mathrm{Gr}_m^{\tau.}(Rj_* V_{A\otimes\mathbb{Q}}), (\mathrm{Gr}_m^{W.}(\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M}), F^\bullet))$$

is a cohomological $A \otimes \mathbb{Q}$-Hodge complex of weight $m$, i.e. the spectral sequence associated to the filtration of the second complex degenerates at $E_1$, and the induced filtration on

$$\mathbb{H}^k(\mathrm{Gr}_m^{W.}(\Omega_X^\bullet\langle D\rangle \otimes \mathcal{M})) \cong \mathbb{H}^k(\mathrm{Gr}_m^{\tau.}(Rj_* V_{A\otimes\mathbb{Q}})) \otimes \mathbb{C}$$

defines a pure $A \otimes \mathbb{Q}$-Hodge structure of weight $m+k$ on $\mathbb{H}^k(\mathrm{Gr}_m^{\tau.}(Rj_* V_{A\otimes\mathbb{Q}}))$.

Now $\quad \mathrm{Gr}_m^{\tau.}(Rj_* V_{A\otimes\mathbb{Q}}) = (\cdots \to 0 \to (Rj_* V_{A\otimes\mathbb{Q}})^{m-1}/\mathrm{Ker}\,\partial^{m-1} \to \mathrm{Ker}\,\partial^m \to 0 \to \cdots).$

Hence $\mathrm{Gr}_m^{\tau.}(Rj_* V_{A\otimes\mathbb{Q}})$ is quasiisomorphic to $R^m j_* V_{A\otimes\mathbb{Q}}[-m]$, which in turn by (3. 2) is quasiisomorphic to $j_{m*} V_{m, A\otimes\mathbb{Q}}(-m)\,[-m]$.

For the second complex, we use the isomorphism (1.7):

$$(\mathrm{Gr}_m^{W\cdot}(\Omega_X^{\cdot}\langle D\rangle \otimes \mathscr{M}), F^{\cdot}) \xrightarrow[\sim]{\mathrm{Res}_m(\mathscr{M})} v_{m*}(\tilde{\Omega}^{\cdot}_{\tilde{D}_m}(\mathscr{M}_m), \tilde{F}^{\cdot})[-m],$$

where $\tilde{F}^{\cdot}(\tilde{\Omega}^{\cdot}_{\tilde{D}_m}(\mathscr{M}_m))$ is given by:

$$\tilde{F}^p(\tilde{\Omega}^q_{\tilde{D}_m}(\mathscr{M}_m)) = \begin{cases} 0, & \text{if } q+m<p \\ \tilde{\Omega}^q_{\tilde{D}_m}(\mathscr{M}_m), & \text{if } q+m \geqq p \end{cases}.$$

Hence we have to show that

$$v_{m*}(\tilde{j}_{m*}(v_m^{-1} V_{m,A\otimes\mathbb{Q}}(-m)), (\tilde{\Omega}^{\cdot}_{\tilde{D}_m}(\mathscr{M}_m), \tilde{F}^{\cdot}))[-m]$$

is a cohomological $A \otimes \mathbb{Q}$-Hodge complex of weight $m$. This follows if

$$(\tilde{j}_{m*}(v_m^{-1} V_{m,A\otimes\mathbb{Q}}), (\tilde{\Omega}^{\cdot}_{\tilde{D}_m}(\mathscr{M}_m), F^{\cdot}))$$

(with $F^{\cdot}$ the usual Hodge filtration) is a cohomological $A \otimes \mathbb{Q}$-Hodge complex of weight 0. But this is just the statement of (5.3), noting that $\tilde{D}_m$ is bimeromorphic to a disjoint union of compact Kähler manifolds by [7], q.e.d.

Let us now consider the functoriality properties and the dependence on the choice of the compactification of the mixed Hodge structure. Clearly it is functorial for maps $\varphi: V \to V'$ between unitary real local systems which respect the hermitian forms. Of course, the same holds if $\varphi$ is the orthogonal projection onto a real subsystem.

**Lemma (6.6).** *Let $U$ be a manifold with an $A$-unitary local system $V$, $A \subseteqq \mathbb{R}$.*

a) *Let $f: U' \to U$ be a holomorphic map. Suppose that there exist compactifications $U' \hookrightarrow X'$, $U \hookrightarrow X$ with manifolds $X$, $X'$ bimeromorphic to compact Kähler manifolds such that $X' \backslash U'$ and $X \backslash U$ are local normal crossing divisors and such that $f$ extends to a meromorphic map $X' \to X$. Then*

$$f^*: H^k(U, V) \to H^k(U', f^{-1}V)$$

*is a morphism of the mixed Hodge structures induced by $X$ and $X'$.*

b) *If $U \hookrightarrow X$ and $U \hookrightarrow X'$ are two compactifications with manifolds $X$, $X'$ bimeromorphic to compact Kähler manifolds such that $X \backslash U$ and $X' \backslash U$ are local normal crossing divisors and the identity map on $U$ extends to a bimeromorphic map $X \to X'$, then the two mixed Hodge structures on $H^k(U, V)$ induced by $X$ and $X'$ agree.*

*Proof.* In case the extended map $\bar{f}: X' \to X$ is holomorphic, a) follows directly from (4.1).

For b), take a third compactification $U \hookrightarrow X''$ with the same properties, dominating both $X$ and $X'$. Then the identity map on $H^k(U, V)$ is at the same time a morphism of mixed Hodge structures for the mixed Hodge structures induced by $X, X''$ and by $X'', X'$. By [3], 2.3.5 it is an isomorphism of all three mixed Hodge structures.

Then the general case of a) follows from the above case and b).

Of course, if $U$ is algebraic, one uses an algebraic compactification of $U$ to get a mixed Hodge structure on $H^k(U, V)$ which does not depend on the choice of an algebraic compactification and is functorial for algebraic maps $f\colon U' \to U$.

Let us now consider the case of push-forwards. Let $f\colon U \to U'$ be a holomorphic map, $V$ a unitary local system on $U$, such that $f_* \mathbb{C}_U$ and $f_* V$ are local systems on $U'$. Then by composing the sesquilinear pairing $f_* V \times f_* V \to f_* \mathbb{C}_U$ with the trace map $f_* \mathbb{C}_U \to \mathbb{C}_{U'}$ (summing over connected components over small open subsets of $U$), $f_* V$ gets a unitary structure.

**Lemma (6.7).** *Let $U$ be a manifold with an $A$-unitary local system $V$ on $U$, $A \subseteqq \mathbb{R}$. Let $f\colon U \to U'$ be a holomorphic map such that $f_* \mathbb{C}_U$ and $f_* V$ are local systems on $U'$. Suppose there exist compactifications $U \hookrightarrow X$, $U' \hookrightarrow X'$, $X, X'$ bimeromorphic to compact Kähler manifolds, $X \backslash U$ and $X' \backslash U'$ local normal crossing divisors, such that $f\colon U \to U'$ extends to a meromorphic map $X \to X'$. Then the map*

$$H^k(U', f_* V) \to H^k(U, V)$$

*is a morphism of the mixed Hodge structures induced by $X'$ and $X$.*

*Proof.* $H^k(U', f_* V) \to H^k(U, V)$ clearly factors into the lifting map

$$H^k(U', f_* V) \to H^k(U, f^{-1} f_* V)$$

which is a homomorphism of mixed Hodge structures by (6.6) and the map

$$H^k(U, f^{-1} f_* V) \to H^k(U, V)$$

induced by $\varphi\colon f^{-1} f_* V \to V$. $\varphi$ factors into the orthogonal projection

$$f^{-1} f_* V \to (\operatorname{Ker} \varphi)^{\perp}$$

and $(\operatorname{Ker} \varphi)^{\perp} \to V$; the latter map respects the hermitian forms on $(\operatorname{Ker} \varphi)^{\perp}$ and $V$. Hence $H^k(U, f^{-1} f_* V) \to H^k(U, V)$ is a morphism of mixed Hodge structures, too.

For example, if $f\colon U \to U'$ is finite and unramified, then the mixed Hodge structure on $H^k(U', f_* \mathbb{C})$ is nothing but the mixed Hodge structure on $H^k(U, \mathbb{C}) \cong H^k(U', f_* \mathbb{C})$.

## § 7. Non-real unitary local systems

We will now drop the assumption that $V$ is defined over a ring $A \subseteqq \mathbb{R}$. Except for the fact that the weight filtration on $H^k(U, V)$ is defined over $A \otimes \mathbb{Q}$, the fact that $V$ might be defined over a proper subring of $\mathbb{C}$ is not important for what follows, so we take $A = \mathbb{C}$.

$H^k(U, V)$ has no canonical real structure, so one cannot expect a mixed Hodge structure on $H^k(U, V)$. But the statements on degenerations of spectral sequences that one deduces from the presence of a mixed Hodge structure still remain true. More precisely, we have:

**Theorem (7.1).** *Let $V$ be a unitary local system on $U$.*

a) *The spectral sequence associated to the Hodge filtration*

$$E_1^{p,q} = H^q(X, \Omega_X^p \langle D \rangle \otimes \mathcal{M}) \Rightarrow H^{p+q}(U, V)$$

*degenerates at $E_1$.*

b) *The Leray spectral sequence*

$$E_2^{p,q} = H^p(X, R^q j_* V) \cong H^p(X, j_{q*} V_q) \Rightarrow H^{p+q}(U, V)$$

*degenerates at* $E_3$.

c) *The spectral sequence associated to the induced weight filtration on* $\Omega_X^p \langle D \rangle \otimes \mathcal{M}$

$$E_1^{-m,k+m} = H^k(X, \mathrm{Gr}_m^{W\cdot}(\Omega_X^p \langle D \rangle \otimes \mathcal{M})) \cong H^k(\tilde{D}_m, \tilde{\Omega}_{\tilde{D}_m}^{p-m}(\mathcal{M}_m)) \Rightarrow H^k(X, \Omega_X^p \langle D \rangle \otimes \mathcal{M})$$

*degenerates at* $E_2$.

d) *The conjugate linear isomorphism* $H^k(U, V) \longrightarrow H^k(U, V^\vee)$ *respects the weight filtrations and induces a conjugate linear isomorphism between the "Hodge components"*

$$\mathrm{Gr}_{F\cdot}^p \mathrm{Gr}_{p+q}^{W\cdot} H^k(U, V) \text{ and } \mathrm{Gr}_{F\cdot}^q \mathrm{Gr}_{p+q}^{W\cdot} H^k(U, V).$$

**Remark.** The quasiisomorphism $R^q j_* V \longrightarrow \mathrm{Gr}_q^{W\cdot}(\Omega_X^\cdot \langle D \rangle \otimes \mathcal{M})[q]$ identifies the Leray spectral sequence with the spectral sequence ${}_W E_1^{p,q} \Rightarrow H^{p+q}(U, V)$ associated to the weight filtration, with the renumbering $E_2^{p,q} \mapsto {}_W E_1^{-q,p+2q}$. Of course, the appearance of negative indices (as well as in c) above) comes from the fact that $W.$ is an increasing filtration.

*Proof.* Let $\langle .\,,.\rangle_V$ be the hermitian inner product on $V$. $\phi: V \longrightarrow V^\vee$ is the conjugate linear isomorphism which sends a germ $v$ in $V$ to the functional $(w \mapsto \langle w, v \rangle_V)$. The inner product on $V^\vee$ is given by $\langle v^*, w^* \rangle_{V^\vee} = \overline{\langle \phi^{-1}(v^*), \phi^{-1}(w^*) \rangle_V}$.

The clue is to look at $V \otimes V^\vee$, with the hermitian inner product

$$\langle (v, v^*), (w, w^*) \rangle_{V \oplus V^\vee} := \langle v, w \rangle_V + \langle v^*, w^* \rangle_{V^\vee}.$$

$V \oplus V^\vee$ has a real structure: Let $(V \oplus V^\vee)_{\mathbb{R}}$ be the real subsystem consisting of all germs $(v, v^*)$ with $v^* = \phi(v)$.

The canonical map $(V \oplus V^\vee)_{\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C} \longrightarrow V \oplus V^\vee$ is an isomorphism, with inverse

$$(v, v^*) \mapsto \left(\frac{1}{2}(v + \phi^{-1}(v^*)), \frac{1}{2}(\phi(v) + v^*)\right) \otimes 1 + \left(\frac{1}{2i}(v - \phi^{-1}(v^*)), -\frac{1}{2i}(\phi(v) - v^*)\right) \otimes i.$$

Furthermore, $\langle (v, \phi(v)), (w, \phi(w)) \rangle = 2\,\mathrm{Re}\langle v, w \rangle$. Hence $V \oplus V^\vee$ is an $\mathbb{R}$-unitary local system.

By (6.3), $H^k(U, V \oplus V^\vee)$ has a mixed $\mathbb{R}$-Hodge structure. Hence, by [3], 3.2.13, a), b) and c) are true for $V \oplus V^\vee$. The logarithmic de Rham complex of $V \oplus V^\vee$ is the direct sum of the logarithmic de Rham complexes of $V$ and $V^\vee$, and by their definitions the same is true for both filtrations. Hence the spectral sequences for $V \oplus V^\vee$ are the direct sums of the spectral sequences for $V$ and $V^\vee$, so these degenerate at the same levels.

Part d) follows from the fact that the conjugation map on $V \oplus V^\vee$ with respect to the real structure on $V \oplus V^\vee$ is given by $(v, v^*) \mapsto (\phi^{-1}(v^*), \phi(v))$. Note that $\phi$ does not induce a map between $F^p \mathrm{Gr}^W_{p+q} H^{p+q}(U, V)$ and $F^q \mathrm{Gr}^W_{p+q} H^{p+q}(U, V^\vee)$. But if $\bar{F}^\cdot$ denotes the filtrations $\phi^{-1}(F^\cdot)$ on $V$ or $\phi(F^\cdot)$ on $V^\vee$, then

$$(F^p \cap \bar{F}^q)\, \mathrm{Gr}^W_{p+q} H^{p+q}(U, V) \cong \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^{p+q}(U, V),$$
$$(F^q \cap \bar{F}^p)\, \mathrm{Gr}^W_{p+q} H^{p+q}(U, V^\vee) \cong \mathrm{Gr}^q_F \mathrm{Gr}^W_{p+q} H^{p+q}(U, V^\vee),$$

and $\phi$ clearly induces a conjugate linear isomorphism between these two spaces.

**Remark (7. 2).** Clearly the functoriality properties (6. 6) and (6. 7) have counterparts in the non-real case, i.e. the lifting and push-forward maps on cohomology respect Hodge and weight filtrations.

**Remark (7. 3).** It is possible to prove (7. 1. a) without using the full strength of Deligne's mixed Hodge theory. Let me sketch how one can directly deduce the degeneration of the Hodge spectral sequence for logarithmic forms.

We compute $H^k(U, V) \cong \mathbb{H}^k(X, \Omega^\cdot_X\langle D\rangle \otimes \mathcal{M})$ by using logarithmic $C^\infty$-forms: Let $A^{p,q}_X\langle D\rangle(V) := \Gamma(X, \Omega^p_X\langle D\rangle \underset{\mathcal{O}_X}{\otimes} \mathcal{E}^{0,q}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M})$, where $\mathcal{E}^{0,q}_X$ is the sheaf of $C^\infty$-$(0,q)$-forms on $X$. With the operatores $\nabla$ and $\bar\partial$, $A^{\cdot\cdot}_X\langle D\rangle(V)$ is a double complex, and $H^k(U, V)$ is the $k$-th cohomology of the associated single complex. Similarly, $H^k(X, j_* V)$ is computed from $\tilde{A}^{p,q}_X(V) := \Gamma(X, \tilde{\Omega}^p_X(\mathcal{M}) \underset{\mathcal{O}_X}{\otimes} \mathcal{E}^{0,q}_X)$.

Let us first show that for $\varphi \in A^{p,q}_X\langle D\rangle(V)$ with $\bar\partial\varphi = 0$, $\nabla\varphi \in \tilde{A}^{p+1,q}_X(V)$, there exists $\psi \in \tilde{A}^{p,q-1}_X(V)$ with $\nabla\varphi = \nabla\bar\partial\psi$.

For that, we use Serre duality: So let $\sigma \in \tilde{A}^{n-p-1,n-q}_X(V^\vee)$ with $\bar\partial\sigma = 0$. Changing $\sigma$ by a $\bar\partial$-exact form, we may assume $\nabla\sigma = 0$. Then

$$\int_X \nabla\varphi \wedge \sigma = \int_X d(\varphi \wedge \sigma) - \int_X \varphi \wedge \nabla\sigma = \int_X d(\varphi \wedge \sigma).$$

Note that logarithmic $(n, n)$-forms are integrable. We have

$$\int_X d(\varphi \wedge \sigma) = -\frac{1}{2\pi i} \int_{\tilde{D}_1} \mathrm{Res}_1(\varphi \wedge \sigma) = 0,$$

because $\mathrm{Res}_1(\varphi \wedge \sigma)$ is an $(n-2, n)$-form on $\tilde{D}_1$, hence 0. So $\nabla\varphi$ is $\bar\partial$-exact, and therefore $\nabla\bar\partial$-exact (see e.g. [3], 4. 3. 1).

For the degeneration of the Hodge spectral sequence, it clearly suffices to show: For $\varphi \in A^{p,q}_X\langle D\rangle(V)$ with $\bar\partial\varphi = 0$ and $\mathrm{Res}_{m+1}(\mathcal{M})(\nabla\varphi) = 0$, there exists $\psi \in A^{p,q-1}_X\langle D\rangle(V)$ with $\mathrm{Res}_m(\mathcal{M})(\nabla(\varphi - \bar\partial\psi)) = 0$ (here $\mathrm{Res}_0(\mathcal{M}) := \mathrm{id}$).

For $\tilde\varphi := \mathrm{Res}_m(\mathcal{M})(\varphi)$ we have $\bar\partial\tilde\varphi = 0$, $\nabla_m\tilde\varphi \in \tilde{A}^{p-m+1,q}_{\tilde{D}_m}(v_m^{-1} V_m)$, so by the above there exists $\tilde\psi \in \tilde{A}^{p-m,q-1}_{\tilde{D}_m}(v_m^{-1} V_m)$ with $\nabla_m\tilde\varphi = \nabla_m\bar\partial\tilde\psi$. Letting $\psi \in A^{p,q-1}_X\langle D\rangle(V)$ with $\mathrm{Res}_m(\mathcal{M})(\psi) = \tilde\psi$, we have

$$\mathrm{Res}_m(\mathcal{M})(\nabla(\varphi - \bar\partial\psi)) = \nabla_m\tilde\varphi - \nabla_m\bar\partial\tilde\psi = 0, \quad \text{q.e.d.}$$

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